10-13-2021

- Homework due monday (10-18-2021)

Last time: 2^{nd} derivative test

$D = f_{xx} f_{yy} - f_{xy}^2$, $\vec{p}$ a critical point.

① $f_{xx}(\vec{p}) > 0$ and $D(\vec{p}) > 0 \rightarrow \vec{p}$ a local min point.

② $f_{xx}(\vec{p}) < 0$ and $D(\vec{p}) > 0 \rightarrow \vec{p}$ a local max point.

③ $D(\vec{p}) < 0 \rightarrow \vec{p}$ a saddle point.

Note: If $D(\vec{p}) = 0$, we cannot conclude anything about $\vec{p}$ using this test.

Note 2: You could use $f_{yy}(\vec{p})$ in place of $f_{xx}(\vec{p})$

EXAMPLE: Classify critical points of $f(x,y) = xy + e^{-xy}$ using 2^{nd} derivative test:

$$\nabla f = \langle y - ye^{-xy}, x - xe^{-xy} \rangle = \langle y(1-e^{-xy}), x(1-e^{-xy}) \rangle$$

$$\nabla f = \vec{0} \text{ if and only if } \begin{cases} y(1-e^{-xy}) = 0 \\ x(1-e^{-xy}) = 0 \end{cases}$$

$$\text{if and only if } \begin{cases} y=0 \text{ or } 1-e^{-xy} = 0 \\ x=0 \text{ or } 1-e^{-xy} = 0 \end{cases}$$

Note: $e^{-xy} = 1$ if and only if $-xy = 0$ if and only if $x=0$ or $y=0$

$\therefore \nabla f = \vec{0}$ if and only if either $x=0$ or $y=0$ b/c $x=0$ or $y=0$ implies both of the conditions for $\nabla f = \vec{0}$

Problem Continued

Now we need $D(x,y)$:

$$f_{xx} = y^2 e^{-xy} \qquad f_{yy} = x^2 e^{-xy}$$

$$f_{xy} = f_{yx} = 1 - \left(e^{-xy} + \left(-xye^{-xy}\right)\right) = 1 - e^{-xy}(1-xy)$$

$$\therefore D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

$$= \left(y^2 e^{-xy}\right)\left(x^2 e^{-xy}\right) - \left(1 - e^{-xy}(1-xy)\right)^2$$

$D(x,y) = 0$ uniformly when $x=0$ or $y=0$

$\therefore$ Nothing can be said via 2^{nd} derivative test.

EXAMPLE: lets classify critical points of $f(x,y) = x^2 + y^2 + xy + y$ via 2^{nd} derivative test

$$\nabla f = \langle 2x+y, 2y+x+1 \rangle$$

$\therefore \nabla f = \vec{0}$ if and only if $\begin{cases} 2x+y=0 \\ 2y+x+1=0 \end{cases}$

if and only if $\begin{cases} y = -2x \\ 2(-2x)+x+1=0 \end{cases}$

if and only if $\begin{cases} -3x+1=0 \\ y=-2x \end{cases}$

if and only if $\begin{cases} x = 1/3 \\ y = -2/3 \end{cases}$

$\therefore$ we have a unique critical point at $\left(\frac{1}{3}, \frac{-2}{3}\right)$

Problem Continued

$$f_{xx} = 2 \ , \ f_{yy} = 2 \ , \ f_{xy} = f_{yx} = 1$$

$\therefore$ $D(x,y) = 2 \cdot 2 - 1^2 = 3 > 0$

$\therefore$ b/c $f_{xx}\left(\frac{1}{3}, -\frac{2}{3}\right) = 2 > 0$ and $D\left(\frac{1}{3}, -\frac{2}{3}\right) = 3 > 0$
we have by 2^{nd} derivative test: $(1/3, -2/3)$ is a local minimum point within local min value

$$f(1/3, -2/3) = (1/3)^2 + (-2/3)^2 + (1/3)(-2/3) + (-2/3)$$

$$= (1/9) + (4/9) - (2/9) - (6/9) = -\frac{1}{3}$$

EXAMPLE: Classify critical points $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

$$\nabla f = \langle 3x^2 - 6x - 9 \ , \ 3y^2 - 6y \rangle$$

$$= 3\langle x^2 - 2x - 3, \ y^2 - 2y \rangle$$

$$= 3\langle (x-3)(x+1), \ y(y-2) \rangle$$

$\therefore$ $\nabla f = \vec{0}$ if and only if $\begin{cases} (x-3)(x+1) = 0 \\ y(y-2) = 0 \end{cases}$

if and only if $\begin{cases} x = 3 \text{ or } x = -1 \\ y = 0 \text{ or } y = -2 \end{cases}$

	$x=3$	$x=-1$
$y=0$	$(3,0)$	$(-1,0)$
$y=2$	$(3,2)$	$(-1,2)$

$\therefore$ we have critical points: $(3,0), (-1,0), (3,2), (-1,2)$

Compute $D(x,y)$

$$f_{xx} = 6x - 6 \qquad f_{yy} = 6y - 6 \qquad f_{xy} = f_{yx} = 0$$
$$= 6(x-1) \qquad\qquad = 6(y-1)$$

Problem Continued

$$\therefore D(x,y) = f_{xx} \cdot f_{yy} - f_{xy}^2 = 6(x-1) \cdot 6(y-1) - 0^2 = 36(x-1)(y-1)$$

for $p = (3,0)$: $D(x,y) = 36(3-1)(0-1) < 0$ and $f_{xx}(\vec{p}) = 6(3-1) > 0$
b/c $D(3,0) < 0$, $(3,0)$ is a saddle point

for $p = (-1,0)$: $D(\vec{p}) = 36(-1-1)(0-1) > 0$ and $f_{xx}(\vec{p}) = 6(-1-1) < 0$
$\therefore$ $(-1,0)$ is a local max point of f w/ a local max value of $f(-1,0) = 5$

for $p = (3,2)$: $D(\vec{p}) = 36(3-1)(2-1) > 0$ and $f_{xx}(\vec{p}) = 6(3-1) > 0$
$\therefore$ $(3,2)$ is a local minimum point of f w/ local min value $f(3,2) = -31$

for $p = (-1,2)$: $D(\vec{p}) = 36(-1-1)(2-1) < 0$
$\therefore$ $(-1,2)$ is a saddle point of f

Lagrange Multipliers

Idea: Method for solving constrained optimization problems

Observation: if $f: \mathbb{R}^n \to \mathbb{R}$ has a level curve $f(\vec{x}) = c$, then for every point $\vec{p}$ on this level curve: $\nabla f(\vec{p}) = \nabla c = \vec{0}$

i.e. ∇f is orthoginal to level curves

So the idea at this point: Set up constrained optimization to be optimization on a level curve...

In general:

$$\begin{cases} \text{optimize } f(\vec{x}) \\ \text{subject to } \underbrace{g_1(\vec{x}) = 0,\ g_2(\vec{x}) = 0}_{\text{zero-level curves}}, \ldots \end{cases}$$

Can be turned into optimize:

$$F(\vec{x}, \lambda_1, \lambda_2, \dots \lambda_k) = f(\vec{x}) - \lambda_1 g_1(\vec{x}) - \lambda_2 g_2(\vec{x}), \dots - \lambda_k g_k(\vec{x})$$

$F(\vec{x}, \vec{\lambda})$ has critical points via $\nabla F = 0$

So supposing the solution set of $g_1(\vec{x}) = 0 = g_2(\vec{x}) = \dots = g_k(\vec{x})$ is closed and bounded, the local extrema of F determine global extrema of f under $g_1 = g_2 = \dots = g_k = 0$

EXAMPLE: Lets optimize $f(x,y) = xe^y$ subject to $\underbrace{x^2+y^2=2}$

Should be satisfied if and only if $g(x,y)=0$

via Lagrange Multipliers:

$$F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

w/ $g(x,y) = x^2+y^2-2$ (b/c $g(x,y)=0$ if and only if $x^2+y^2=2$)

So, $F(x,y,\lambda) = xe^y - \lambda(x^2+y^2-2)$

$\therefore \nabla F = \langle e^y - \lambda(2x),\ xe^y - \lambda(2y),\ x^2+y^2-2 \rangle$

So, $\nabla F = 0$ if and only if $\begin{cases} e^y - \lambda 2x = 0 \\ xe^y - \lambda 2y = 0 \\ -x^2 - y^2 + 2 = 0 \end{cases}$

if and only if $\begin{cases} 2\lambda x = e^y & (1) \\ 2\lambda y = xe^y & (2) \\ -x^2 - y^2 = -2 & (3) \end{cases}$

(Note: solve this system)

$2\lambda x = e^y$ implies $\lambda \neq 0$ (unless $e^y = 0$, which is nonsense!)

with equation 1 and 2 we obtain the following: $2\lambda y = x(2\lambda x)$

$\therefore 2\lambda y = 2\lambda x^2$ Now $\lambda \neq 0$ yeilds $y = x^2$

So, equation 3 becomes $x^2 + (x^2)^2 = 2$

i.e. $(x^2)^2 + (x^2) - 2 = 0$

i.e. $(x^2+2)(x^2-1) = 0$

i.e. $(x^2+2)(x-1)(x+1) = 0$

$\therefore x = 1$ or $x = -1$

if $x = -1$ then $y = (-1)^2$ and $\lambda = \frac{e^y}{2x} = \frac{e^1}{2(-1)} = -\frac{e}{2}$

So, $(-1, 1)$ is a potential extreme point of f subject to $g = 0$

Note: Lowercase f does not care about λ

$f(-1,1) = (-1)e^1 = -e$

$x = -1$: $y = 1^2 = 1$, $\lambda = \frac{e^1}{2 \cdot 1} = \frac{e}{2}$

$\therefore (1,1)$ is also a possible extreme point

$f(1,1) = 1 \cdot e^1 = e$

$\therefore -e$ is the global min and e is the global max of f subject to $x^2 + y^2 = 2$